Characteristic polynomial, eigenvalues, and eigenvectors 3

For each of the following matrices, we request:

1. Find the characteristic polynomial.
2. Find the eigenvalues and the associated eigenvectors.

f)

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

g)

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

h)

$$\begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}$$

Solution

f) Characteristic polynomial

Let

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}.$$

To find the characteristic polynomial of A, we compute

$$p(\lambda) = \det(A - \lambda I).$$

In this case,

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{pmatrix}.$$

Its determinant is:

$$(3-\lambda)(2-\lambda) - (-2)(-1) \;=\; (3-\lambda)(2-\lambda) - 2.$$

Expanding,

$$(3-\lambda)(2-\lambda) = 6 - 3\lambda - 2\lambda + \lambda^2 = \lambda^2 - 5\lambda + 6,$$

so that

$$p(\lambda) = \lambda^2 - 5\lambda + 6 \;-\; 2 = \lambda^2 - 5\lambda + 4.$$

Therefore, the characteristic polynomial of A is:

$$p(\lambda) = \lambda^2 - 5\lambda + 4.$$

Eigenvalues and eigenvectors

The eigenvalues are the roots of $p(\lambda)$:

$$\lambda^2 - 5\lambda + 4 = 0 \quad \Longrightarrow \quad (\lambda-4)(\lambda-1) = 0,$$

so that

$$\lambda_1 = 4, \quad \lambda_2 = 1.$$

Eigenvalue $\lambda_1 = 4$

To find the associated eigenvector, we solve

$$(A - 4I)\,\mathbf{v} = 0.$$

In matrix form:

$$A - 4I = \begin{pmatrix} 3-4 & -2 \\ -1 & 2-4 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}.$$

The resulting linear system is:

$$\begin{cases} -x \;-\; 2y = 0, \\ -x \;-\; 2y = 0. \end{cases}$$

Both equations are equivalent to $-x - 2y = 0$, that is, $x = -2y$. Choosing $y = 1$, we obtain $x = -2$. An associated eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda_2 = 1$

For this eigenvalue, we solve

$$(A - I)\,\mathbf{v} = 0.$$

In matrix form:

$$A - I = \begin{pmatrix} 3-1 & -2 \\ -1 & 2-1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$

The resulting linear system is:

$$\begin{cases} 2x \ - \ 2y = 0, \\ -x \ + \ y = 0. \end{cases}$$

Both equations imply $x = y$. Choosing $x = 1$, it follows that $y = 1$. An associated eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

g) Characteristic polynomial

Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$$

To find the characteristic polynomial of A, we compute

$$p(\lambda) = \det(A - \lambda I).$$

We have:

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & -1 \\ 2 & 2-\lambda & 1 \\ 2 & -1 & 4-\lambda \end{pmatrix}.$$

Using cofactor expansion, we obtain:

$$\begin{aligned} p(\lambda) &= (1-\lambda)\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 2 & 1 \\ 2 & 4-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 2 & 2-\lambda \\ 2 & -1 \end{pmatrix} \\ &= (1-\lambda)\Big[(2-\lambda)(4-\lambda)+1\Big] - \Big[2(4-\lambda)-2\Big] - \Big[2(-1)-2(2-\lambda)\Big]. \end{aligned}$$

We compute:

$$(2-\lambda)(4-\lambda) = \lambda^2 - 6\lambda + 8,$$

so that:

$$(2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2.$$

Additionally,

$$2(4-\lambda) - 2 = 8 - 2\lambda - 2 = 6 - 2\lambda,$$

and

$$2(-1) - 2(2-\lambda) = -2 - 4 + 2\lambda = 2\lambda - 6.$$

Thus,

$$p(\lambda) = (1-\lambda)(\lambda-3)^2 - (6-2\lambda) - (2\lambda-6).$$

Since

$$-(6-2\lambda) - (2\lambda-6) = -6 + 2\lambda - 2\lambda + 6 = 0,$$

we obtain

$$p(\lambda) = (1-\lambda)(\lambda-3)^2.$$

That is, the characteristic polynomial is:

$$p(\lambda) = (1-\lambda)(\lambda-3)^2.$$

Eigenvalues and eigenvectors

The eigenvalues are the roots of $p(\lambda) = 0$:

$$1-\lambda = 0 \quad \Rightarrow \quad \lambda = 1,$$

$$(\lambda-3)^2 = 0 \quad \Rightarrow \quad \lambda = 3 \quad \text{(double multiplicity)}.$$

Eigenvalue $\lambda_1 = 1$

To find the eigenvector associated with $\lambda = 1$, we solve:

$$(A - I)\mathbf{v} = 0.$$

We have:

$$A - I = \begin{pmatrix} 1-1 & 1 & -1 \\ 2 & 2-1 & 1 \\ 2 & -1 & 4-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & 3 \end{pmatrix}.$$

This leads to the system:

$$\begin{cases} y - z = 0, \\ 2x + y + z = 0, \\ 2x - y + 3z = 0. \end{cases}$$

From the first equation, we deduce $y = z$. Substituting into the second:

$$2x + 2y = 0 \quad \Longrightarrow \quad x = -y.$$

The third equation is automatically satisfied. Choosing $y = 1$, we obtain $x = -1$ and $z = 1$. Thus, an associated eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda_2 = 3$

For $\lambda = 3$, we solve:

$$(A - 3I)\mathbf{v} = 0.$$

We have:

$$A - 3I = \begin{pmatrix} 1-3 & 1 & -1 \\ 2 & 2-3 & 1 \\ 2 & -1 & 4-3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

The resulting system is:

$$\begin{cases} -2x + y - z = 0, \\ 2x - y + z = 0, \\ 2x - y + z = 0. \end{cases}$$

The first equation implies:

$$-2x + y - z = 0 \quad \Longrightarrow \quad y = 2x + z.$$

Thus, the general solution can be written as:

$$\mathbf{v} = \begin{pmatrix} x \\ 2x + z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

A possible choice of eigenvectors, consistent with those found in other sources, is:

$$\mathbf{v}_2 = \begin{pmatrix} 0.5 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} -0.5 \\ 0 \\ 1 \end{pmatrix}.$$

These vectors, being defined up to a multiplicative factor, form a basis for the eigenspace associated with $\lambda = 3$.

h) Characteristic polynomial

Let

$$A = \begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}.$$

To find the characteristic polynomial of A, we compute

$$p(\lambda) = \det(A - \lambda I),$$

where

$$A - \lambda I = \begin{pmatrix} -5-\lambda & -5 & -9 \\ 8 & 9-\lambda & 18 \\ -2 & -3 & -7-\lambda \end{pmatrix}.$$

Performing the necessary computations, we obtain:

$$p(\lambda) = -(\lambda + 1)^3.$$

Multiplying by -1, we obtain the monic polynomial:

$$p(\lambda) = (\lambda + 1)^3.$$

Eigenvalues and eigenvectors

The equation $p(\lambda) = 0$ simplifies to

$$(\lambda + 1)^3 = 0,$$

which gives the only eigenvalue:

$$\lambda = -1 \quad \text{(with algebraic multiplicity 3)}.$$